We want to fit a poly^{n} of deg. m of the form: $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$

with using observed values:

(x_j, y_j), $j = 1(1)n$.

Let $y = Y_j$, expected value of y at $x = x_j$.

$\Rightarrow$ Residue $= (y_j - Y_j)$

Let $S^2 = \sum_{1}^{n} (y_j - Y_j)^2$ ← sum of squares of residuals

$= \sum (y_j - a_0 - a_1 x_j - a_2 x_j^2 - \cdots - a_m x_j^m)^2$

We minimize S^2 by method of least squares:

i) $\frac{\partial S^2}{\partial a_0} = 0 \rightarrow \sum y_j = n a_0 + a_1 \sum x_j + a_2 \sum x_j^2 + \cdots + a_m \sum x_j^m$

ii) $\frac{\partial S^2}{\partial a_1} = 0 \rightarrow \sum x_j y_j = a_0 \sum x_j + a_1 \sum x_j^2 + \cdots + a_m \sum x_j^{m+1}$

iii) $\frac{\partial S^2}{\partial a_2} = 0 \rightarrow \sum x_j^2 y_j = a_0 \sum x_j^2 + a_1 \sum x_j^3 + \cdots + a_m \sum x_j^{m+2}$

$\vdots$

m+1) $\frac{\partial S^2}{\partial a_m} = 0 \rightarrow \sum x_j^m y_j = a_0 \sum x_j^m + a_1 \sum x_j^{m+1} + \cdots + a_m \sum x_j^{m+m}$

↳ Normal Equations

1) Use the least square method to fit the line $y = a + bx$ based on the observed values $(2, 1)$, $(\frac{1}{6}, -\frac{5}{6})$, $(-\frac{3}{2}, -2)$ & $(-\frac{1}{3}, -\frac{2}{3})$.

Ans: $y = -0.6943 + 0.8319x$ (Write the program also)

i) Fitting a straight line : $y = a + bx$

$$S^2 = \sum_1^n \left[y_j - (a + bx_j)\right]^2$$

Sol^ns: $a = \frac{1}{n}\left[\sum y_j - b \sum x_j\right]$,

$$\frac{\partial S^2}{\partial a} = 0 \Rightarrow \sum y_j = na + b \sum x_j$$

$$\frac{\partial S^2}{\partial b} = 0 \Rightarrow \sum x_j y_j = a \sum x_j + b \sum x_j^2$$

$$b = \frac{n \sum x_j y_j - \sum x_j \sum y_j}{n \sum x_j^2 - \left(\sum x_j\right)^2}$$

Solve for a & b. & put in $y = a + bx$.

ii) $y = a + bx + cx^2 \longrightarrow$ (*)

(x_j, y_j), $j = 1(1)n \longrightarrow n$ obs. of (x, y).

$y = a + bx + cx^2$ be the 2nd deg. pol^n to be fitted on the basis of the observed data.

Residuals at $x = x_j$ is $y_j - Y_j$

$$= y_j - (a + bx_j + cx_j^2)$$

$$\Rightarrow S^2 = \sum \left(y_j - a - bx_j - cx_j^2\right)^2$$

Normal Eq^ns
$$\sum y_j = na + b \sum x_j + c \sum x_j^2$$
$$\sum x_j y_j = a \sum x_j + b \sum x_j^2 + c \sum x_j^3$$
$$\sum x_j^2 y_j = a \sum x_j^2 + b \sum x_j^3 + c \sum x_j^4$$

Solve for a, b & c and put in (*)

iii) $y = a + bx + cx^2 + dx^3$ (Cubic pol^n)

$$S^2 = \sum \left(y_j - a - bx_j - cx_j^2 - dx_j^3\right)^2$$

NE:
$$\sum y_j = na + b \sum x_j + c \sum x_j^2 + d \sum x_j^3$$
$$\sum x_j y_j = a \sum x_j + b \sum x_j^2 + c \sum x_j^3 + d \sum x_j^4$$

Solve
$$\sum x_j^2 y_j =$$
$$\sum x_j^3 y_j =$$

Let the ~~fitted~~ cubic poly^n to be fitted is

$v = a + bu + cu^2 + du^3$,

$u = (x-a')/c'$, $v = (y-b')/d'$

(a', b', c', d' : suitable constants, with $c' \neq 0$, $d' \neq 0$)

$\sum v = na + b\sum u + c\sum u^2 + d\sum u^3$

$\sum uv =$

$\sum u^2 v =$

$\sum u^3 v =$

Solve for a, b, c & d and we have

$v = a + bu + cu^2 + du^3$

$$\left(\frac{y-b'}{d'}\right) = a + b\left(\frac{x-a'}{c'}\right) + c\left(\frac{x-a'}{c'}\right)^2 + d\left(\frac{x-a'}{c'}\right)^3$$

$\Rightarrow y = a'' + b''x + c''x^2 + d''x^3$, ~~reqd~~ required cubic polynomial

2. Fit a parabola to the following data:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Let the parabola to be fitted is

$v = a + bu + cu^2$,

$u = x - 5$, $v = y - 10$

solve for a, b, c:

$\sum v = na + b\sum u + c\sum u^2$ ($n = 9$)

$\sum uv =$

$\sum u^2 v =$

Ans: $y = -0.9282 + 3.5230x - 0.2673x^2$

Fitting a Polynomial of degree one: y = a + bx

Algorithm

---

1. Start

2. Read the Number of observed data (n)

3. For i=1 to n:
   Read X_i and Y_i
   Next i

4. Initialize:
   sumX = 0
   sumX2 = 0
   sumY = 0
   sumXY = 0

5. Calculate the Required Sum
   For i=1 to n:
   sumX = sumX + X_i
   sumX2 = sumX2 + X_i * X_i
   sumY = sumY + Y_i
   sumXY = sumXY + X_i * Y_i
   Next i

6. Calculate the Required Constants a and b of y = a + bx:
   b = (n * sumXY - sumX * sumY)/(n*sumX2 - sumX * sumX)
   a = (sumY - b*sumX)/n

7. Display values of a and b

8. Stop

C Program

```
#include<stdio.h>
int main()
{
 int n, i;
 float x[100], y[100], sumX=0, sumX2=0, sumY=0, sumXY=0, a, b;
printf("The number of observed data: \n");
 scanf("%d", &n);
 printf("Enter data:  \n");
 for(i=1;i<=n;i++)
 {
  printf("x[%d]=",i);
  scanf("%f", &x[i]);
  printf("y[%d]=",i);
  scanf("%f", &y[i]);
 }
 for(i=1;i<=n;i++)
 {
  sumX = sumX + x[i];
  sumX2 = sumX2 + x[i]*x[i];
  sumY = sumY + y[i];
  sumXY = sumXY + x[i]*y[i];
 }
b = (n*sumXY-sumX*sumY)/(n*sumX2-sumX*sumX);
 a = (sumY - b*sumX)/n;
printf("Values are: a=%0.2f and b = %0.2f",a,b);
 printf("\n Equation of best fit straight line is: y = %0.2f + %0.2fx",a,b);
return(0);
}

//Run the program and try to generalize it for polynomials of degree 2
//and 3
```

3.7 The Remainder (Error) Terms in Interpolation

Let $y = f(x)$ be a function known at $(n + 1)$ distinct arguments x_j and let $y_j = f(x_j)$ be their corresponding entries, $j = 0(1)n$.

Let $\phi(x)$ or $L_n(x)$ be the interpolating polynomial (of degree $\leq n$) of $f(x)$ interpolating at the arguments $x_j, j = 0(1)n$.

So $L_n(x)$ coincides with $f(x)$ only at the nodes x_j, and $L_n(x)$ differs from $f(x)$ for $x \neq x_j$, $j = 0(1)n$. Thus

$$f(x) - \phi(x) = f(x) - L_n(x) \begin{cases} = 0, \text{ for } x = x_j \text{ and} \\ \neq 0, \text{ for } x \neq x_j, j = 0(1)n. \end{cases}$$

Hence if $x \neq x_0, x_1, x_2, ..., x_n$, we always commit some error to approximate the function $f(x)$ by $\phi(x)$ or $L_n(x)$. We denote the error by $E(x)$ or R_n. So $E(x) = R_n = f(x) - \phi(x)$. We now provide a formula of error term or remainder term consisting of the function $f(x)$, its derivatives, all nodes or arguments $x_j, j = 0(1)n$, by means of the following theorem.

■ **Statement : If $f(x)$ be a continuous function and has continuous derivatives upto order $(n + 1)$ in an interval containing the interpolating points : $x_0, x_1, x_2, ..., x_n$ $(x_0 < x_1 < x_2 < ... < x_n)$, then at any point in $x \neq x_j, j = 0(1)n$, the error term or remainder term $E(x)$ in approximating $f(x)$ by the interpolating polynomial $\phi(x)$ (of degree $\leq n$) is given by**

$$E(x) = f(x) - \phi(x) = (x - x_0)(x - x_1) ... (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}, \text{ where } \xi \in (x_0, x_n) \quad ... (8)$$

◆ **Proof** : Let $f(x) = \phi(x) + E(x)$, such that $f(x_j) = \phi(x_j)$, $j = 0(1)n$ and $E(x)$ be the associated error in approximating $f(x)$ by $\phi(x)$.

Let us consider an auxilary function $G(t)$ with independent variable t such that

$$G(t) = [f(t) - \phi(t)] - [f(x) - \phi(x)] \times \frac{(t - x_0)(t - x_1)... (t - x_n)}{(x - x_0)(x - x_1) ... (x - x_n)} \quad ... (9)$$

Clearly,

(i) $G(t)$ vanishes at $(n + 2)$ values of t, *viz.*, $t = x_0, x_1, x_2, ..., x_n$ and x in the given interval.

(ii) $G(t)$ is continuously differentiate upto $(n + 1)$ times in the given interval.

[Since $(t - x_0)(t - x_1) ... (t - x_n)$ and $\phi(t)$ are polynomials and hence they are continuously differentiable, also $f(t)$ is continuously differentiable upto $(n + 1)$ times in the given interval by hypothesis]

Thus by repeated application of Rolle's theorem on the function G(t) in $[x_0, x_1]$, we have

$$G^{(n+1)}(\xi) = 0, \xi \in (x_0, x_n) \quad ... (10)$$

Now we have

(i) $\phi(t)$ is a polynomial of degree $\leq n \Rightarrow \phi^{(n+1)}(\xi) = 0$, also

(ii) $(t - x_0)(t - x_1) ... (t - x_n)$ is a polynomial of degree $(n + 1)$ with coefficient of leading term t^{n+1} is one. Thus its $(n + 1)$th derivative is $(n + 1)!$. Thus differentiating (9) $(n + 1)$ times both sides, and using (10) we get with respect to t

$$0 = G^{(n+1)}(\xi) = [f^{(n+1)}(\xi) - 0] - [f(x) - \phi(x)] \times \frac{(n+1)!}{(x - x_0)(x - x_1) ...(x - x_n)}$$

$$\Rightarrow f(x) - \phi(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

Hence the remainder or error term in interpolation is

$$f(x) - \phi(x) = E(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \textbf{[Proved]}.$$

Notes

1. The error or remainder term in interpolation has not much practical use due to the presence of almost indeterminate term $f^{(n+1)}((\xi)$. If the error is large, the interpolating polynomial $\phi(x)$ [approximation of $f(x)$] is of no use.
2. In most cases the analytical term of $f(x)$ remains unknown. So we can not determine the value of $f^{(n+1)}(\xi)$ involved in the remainder term.
3. Even if somehow the function $f(x)$ is known analytically, still we can not determine the term $f^{(n+1)}(\xi)$, due to very complicated nature of the function.
4. The number $\xi \in (x_0, x_n)$ is unknown and depends on x. So determination of $f^{(n+1)}(\xi)$ is impossible, even if we know $f^{(n+1)}(x)$.

   Hence in any of the situations, the remainder or error term $E(x)$ can not be evaluated exactly. Our objective is to find out its maximum value, *i.e.*, to find out an upper bound of the error term in some special cases only.

Bound of Error

If (i) $|f^{(n+1)}(\xi)| \leq M$ in the given interval $[x_0, x_n]$ and

(ii) $\max\limits_{x \in [x_0, x_n]} |(x - x_0)(x - x_1) \ldots (x - x_n)| = N$, then $|E(x)| \leq \frac{MN}{(n+1)!}$ ... (11)

i.e., $\frac{MN}{(n+1)!}$ is an upper bound of the error $E(x)$.

3.8 Remainder Term and its Bound in Linear Interpolation

In linear interpolation we have two arguments x_0, x_1 and the corresponding entries are $y_0 = f(x_0)$, $y_1 = f(x_1)$ for a continuous differentiable function (upto order two) $y = f(x)$. Let $\phi(x)$ be the interpolating polynomial (of degree ≤ 1). Thus the error or remainder term is given by

$$E_1(x) = f(x) - \phi(x) \begin{cases} = 0, \text{ for } x = x_0, x_1 \\ \neq 0, \text{ for } x \neq x_0, x_1 \end{cases}$$

Consider the auxiliary function $g(t)$ as $g(t) = [f(t) - \phi(t)] - [f(x) - \phi(x)] \times \frac{(t - x_0)(t - x_1)}{(x - x_0)(x - x_1)}$.

It is clear that $g(t) = 0$ at $t = x_0, x_1$ and x. Differentiating $g(t)$ twice with respect to t, we get

$$g''(t) = f''(t) - [f(x) - \phi(x)] \times \frac{2}{(x - x_0)(x - x_1)}.$$

As before by repeated application of Rolle's theorem we have

$g''(\xi) = 0, \xi \in (x_0, x_1)$ [since $\phi(t)$ and $(t - x_0)(t - x_1)$ are polynomials, they are continuously differentiable and such is true for $f(x)$ also by hypothesis]

$$\Rightarrow f(x) = \phi(x) + \frac{1}{2}(x - x_0)(x - x_1)f''(\xi).$$

Thus the remainder term or error term in linear interpolation is given by

$$\mathbf{E_1(x) = f(x) - \phi(x) = \frac{1}{2}(x - x_0)(x - x_1)f''(\xi).} \quad \ldots (12)$$

■ Bound of Error in Linear Interpolation

(i) Let $|f''(\xi)| \le M, \xi \in [x_0, x_1]$ and

(ii) we know the maximum value of $|(x - x_0)(x - x_1)|$ occurs at $x = \frac{1}{2}(x_0 + x_1)$.

Hence maximum value of $|(x - x_0)(x - x_1)|$ is $\frac{1}{4}(x_1 - x_0)^2$.

Therefore from (11) we have,

$$|E_1(x)| \le \frac{1}{2} \max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)| \quad [\text{since } |f''(\xi)| \le M]$$

$$\Rightarrow \mathbf{|E_1(x)| \le \frac{M}{8}(x_1 - x_0)^2} \quad \ldots (13)$$

Note

The error will be minimum, if the arguments x_1, x_0 are close to each other.

5. A function $f(x)$ defined on [0, 1] is such that $f(0) = f(1) = 0$ and $f\left(\frac{1}{2}\right) = -1$. Find the quadratic polynomial $\phi(x)$ which agrees with $f(x)$ for $x = 0, \frac{1}{2}, 1$. If $\left|\frac{d^3f}{dx^3}\right| \le 1$ for $0 \le x \le 1$, show that $|f(x) - \phi(x)| \le \frac{1}{12}$ for $0 \le x \le 1$. [CH 1984]

Solution Here we use the Lagrange's interpolation formula to obtain the quadratic polynomial $\phi(x)$ as

$$\phi(x) = \frac{\left(x-\frac{1}{2}\right)(x-1)}{\left(0-\frac{1}{2}\right)(0-1)} \times 0 + \frac{(x-0)(x-1)}{\left(\frac{1}{2}-0\right)\left(\frac{1}{2}-1\right)} \times (-1) + \frac{(x-0)\left(x-\frac{1}{2}\right)}{(1-0)\left(1-\frac{1}{2}\right)} \times 0$$

$$\Rightarrow \phi(x) = 4x(x-1).$$

The associated error E(x) is given by

$$E(x) = f(x) - \phi(x) = (x-0)\left(x-\frac{1}{2}\right)(x-1)\frac{f^{(3)}(\xi)}{3!}$$

$$\Rightarrow |E(x)| = |f(x) - \phi(x)| = |x-0|\,\left|x-\frac{1}{2}\right|\,|x-1| \times \frac{1}{6}\,|f^{(3)}(\xi)|.$$

$$\le 1 \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{6} \cdot 1 \quad \left[\text{since as } x \in [0, 1],\ |x-0| \le 1,\ \left|x-\frac{1}{2}\right| \le \frac{1}{2},\ |x-1| \le 1 \text{ and } |f^{(3)}(\xi)| \le 1\right]$$

$$= \frac{1}{12}$$

Therefore, $|f(x) - \phi(x)| \le \frac{1}{12}$ **[Proved]**